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Coefficients of fractional parentage for a double-l boson system

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Abstract. The generalised coefficients of fractional parentage (CFP) for a double-l boson system are defined. The formulae of CFP with seniority are presented when the states of this system are classified according to two group chains. The calculations are done in the scheme of second quantisation. The results show that the computation of CFP in this way is very efficient.

1. Introduction

Coefficients of fractional parentage (CFP) have extensive applications in many different fields of microphysics in constructing the many-body wavefunctions. They were first introduced by Bacher and Goudsmit [1]. Since then many people have made important developments [2-5] in the algorithm for CFP calculation and in numerical computation. In our previous paper [6] the CFP for a single-l boson system are factorised into isoscalar factors (ISF) of the symmetry group of this system. A new recursion relation of CFP with well defined seniority provides a faster mechanism for computation of CFP than others without seniority.

Since CFP depend on the classification of states, they are different for differential symmetry group chains of n boson system. For a double-l boson system the symmetry group is unitary group $U(N)$, where $N=N_1+N_2=(2l_1+1)+(2l_2+1)$. In this paper we will discuss two group chains of $U(N)$ including $SO(3)$ as a subgroup. They are

chain I $\quad U(N)\supset U(N_1)\otimes U(N_2)\supset O(N_1)\otimes O(N_2)\supset SO(3_1)\otimes SO(3_2)\supset SO(3)$

chain II $\quad U(N)\supset O(N)\supset O(N_1)\otimes O(N_2)\supset SO(3_1)\otimes SO(3_2)\supset SO(3).$

The generators of groups in these two group chains are presented in table 1.

Chain I represents the coupling between l_1 bosons and l_2 bosons being at the O(3) level (weak coupling). Only the l_i bosons themselves form the l_i pairs. Chain II represents strong coupling, so the generalised pairs coupled by bosons with different l also exist. The U(5) limit and O(6) limit in the IBM model, respectively, provide good examples for each situation.

In section 2 we define the CFP for a double-l boson system in second quantisation. The CFP according to chain I are presented in section 3. In section 4 we construct the

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Table 1. Generators of groups in chain I and chain II.

Group	Generators	
U(N)	$B(ll')^k_q = \{b^\dagger_l b^\dagger_{l'}\}^k_q = \sum_m \langle l\ m\ l'\ m' \mid k\ q\rangle b^\dagger_{lm} b^\dagger_{l'm'}$	$l, l' = l_1, l_2$, all k
O(N)	$O(ll')^k_q = \mathrm{i}B(ll')^k_q - \mathrm{i}(-1)^{l+l'+k} B(l'l)^k_q$	
U(N_1)	$B(l_1 l_1)^k_q$	
U(N_2)	$B(l_2 l_2)^k_q$	
O(N_1)	$O(l_1 l_1)^k_q = 2\mathrm{i}B(l_1 l_1)^k_q$	$k = 2l_1 - 1, 2l_1 - 3, \ldots, 1$
O(N_2)	$O(l_2 l_2)^k_q = 2\mathrm{i}B(l_2 l_2)^k_q$	$k = 2l_2 - 1, 2l_2 - 3, \ldots, 1$
SO(3_1)	$L_{1q} = \left(\frac{l_1(l_1+1)(2l_1+1)}{3}\right)^{1/2} B(l_1 l_1)^l_q$	
SO(3_2)	$L_{2q} = \left(\frac{l_2(l_2+1)(2l_2+1)}{3}\right)^{1/2} B(l_2 l_2)^l_q$	
SO(3)	$L_q = L_{1q} + L_{2q}$	

generalised pair operator and then have the state vectors classified by chain II. The CFP according to chain II are presented in section 5.

2. Definition of the generalised CFP

Let us consider a system of bosons, each with angular momentum l_1 or l_2. In the second quantisation scheme, $b^\dagger_{lm}$ and b_{lm} are boson creation and annihilation operators satisfying the following commutation relation:

$$[b_{lm}, b_{l'm'}] = [b^\dagger_{lm}, b^\dagger_{l'm'}] = 0$$
$$[b_{lm}, b^\dagger_{l'm'}] = \delta_{ll'}\delta_{mm'}. \tag{2.1}$$

For a system of n bosons the state vector with total angular momentum L and its Z component M is $|n\,\alpha\,L\,M\rangle$, where α symbolises additional quantum numbers to completely label the states. The generalised CFP may then be defined as

$$\langle n\ \alpha\ L\{|n-1\ \alpha'\ L'\ l\ L\rangle = \frac{\langle n\ \alpha\ L\|b^\dagger_l\|n-1\ \alpha'\ L'\rangle}{\sqrt{n}} \tag{2.2}$$

$$\langle n\ \alpha\ L\{|n-2\ \alpha''\ L''\ (ll')\ k\ L\rangle = \frac{\langle n\ \alpha\ L\|(b^\dagger_l b^\dagger_{l'})^k\|n-2\ \alpha''\ L''\rangle}{\sqrt{n(n-1)}}. \tag{2.3}$$

These two definitions are similar to that for a single-l boson system in [6] except that $l, l' = l_1, l_2$ and L, L', L'' may be coupled by different l.

It is well known that the state vectors expressed in CFP make the calculation of matrix elements of a one-body tensor operator and a two-body scalar operator simple. It is also known that the two-particle CFP can be expressed in terms of the one-particle CFP. So the key to the question is to construct the one-particle CFP defined in (2.2).

3. CFP according to the group chain I

Chain I provides a set of quantum numbers n, n_1, n_2, σ_1, σ_2, L_1, L_2 and L labelling the irreducible representations (IR) of group U(N), U(N_1), U(N_2), O(N_1), O(N_2), SO(3_1), SO(3_2) and SO(3), respectively. Thus the state vector $|n\,\alpha\,L\,M\rangle$ can now be expressed as

$$|n\ \overbrace{n_1\sigma_1\alpha_1L_1\ \ n_2\sigma_2\alpha_2L_2}^{\alpha}\ L\ M\rangle=\{|n_1\sigma_1\alpha_1L_1\rangle|n_2\sigma_2\alpha_2L_2\rangle\}^L_M \tag{3.1}$$

where $|n_i\,\sigma_i\,\alpha_i\,L_i\,M_i\rangle$ are the known state vectors of identical l_i bosons [6]. By means of the reduction of matrices for a composite system [7] we obtain the CFP according to chain I depending directly on the known CFP of identical bosons as follows:

$$\begin{aligned}&\langle n\ n_1\sigma_1\alpha_1L_1\ n_2\sigma_2\alpha_2L_2\ L\{|n-1\ n_1-1\ \sigma_1'\alpha_1'L_1'\ n_2\sigma_2\alpha_2L_2\ L'\ l_1\ L\rangle\\ &\quad=(n_1/n)^{1/2}\langle n_1\sigma_1\alpha_1L_1\{|n_1-1\ \sigma_1'\alpha_1'L_1'\ l_1\ L_1\rangle\\ &\quad\times\langle(L_1'l_1)\ L_1\ L_2\ L|(L_1'L_2)\ L'\ l_1\ L\rangle\end{aligned} \tag{3.2a}$$

$$\begin{aligned}&\langle n\ n_1\sigma_1\alpha_1L_1\ n_2\sigma_2\alpha_2L_2\ L\{|n-1\ n_1\sigma_1\alpha_1L_1\ n_2-1\ \sigma_2'\alpha_2'L_2'\ L'\ l_2\ L\rangle\\ &\quad=(n_2/n)^{1/2}\langle n_2\sigma_2\alpha_2L_2\{|n_2-1\ \sigma_2'\alpha_2'L_2'\ l_2\ L_2\rangle\\ &\quad+\langle L_1\ (L_2'l_2)\ L_2\ L|(L_1L_2')\ L'\ l_2\ L\rangle\end{aligned} \tag{3.2b}$$

where $\langle\,|\,\rangle$ are the normalised Racah coefficients.

On the other hand, it is straightforward to prove that the operators $b^\dagger_{l_im_i}$, $i=1,2$, form a rank-1 tensor operator of U(N). Its components can be labelled by chain I, namely

$$b^\dagger_{lm}=\begin{cases}b^\dagger_{l_1m_1}=b^\dagger_{[1]\ [\![1]\!]\otimes[\![0]\!]\ (1)\otimes(0)\ l_1\otimes0\ m_1\otimes0}\\ b^\dagger_{l_2m_2}=b^\dagger_{[1]\ [\![0]\!]\otimes[\![1]\!]\ (0)\otimes(1)\ 0\otimes l_2\ 0\otimes m_2}.\end{cases} \tag{3.3}$$

Using the generalised Wigner–Eckart theorem [8], we have

$$\begin{aligned}&\langle n\ n_1\sigma_1\alpha_1L_1\ n_2\sigma_2\alpha_2L_2\ L\|b^\dagger_{l_1}\|n-1\ n_1-1\ \sigma_1'\alpha_1'L_1'\ n_2\sigma_2\alpha_2L_2\ L'\rangle\\ &\quad=\langle n\|b^\dagger\|n-1\rangle\begin{bmatrix}[1] & [n-1] & [n]\\ [\![1]\!]\ [\![0]\!] & [\![n_1-1]\!]\ [\![n_2]\!] & [\![n_1]\!]\ [\![n_2]\!]\end{bmatrix}\\ &\quad\times\begin{bmatrix}[\![1]\!] & [\![n_1-1]\!] & [\![n_1]\!]\\ (1) & (\sigma_1') & (\sigma_1)\end{bmatrix}\begin{bmatrix}(1) & (\sigma_1') & (\sigma_1)\\ l_1 & \alpha_1'L_1' & \alpha_1L_1\end{bmatrix}\\ &\quad\times\langle(L_1'l_1)\ L_1\ L_2\ L|(L_1'L_2)\ L'\ l_1\ L\rangle.\end{aligned} \tag{3.4}$$

Similarly we may have the reduced matrix element of $b^\dagger_{l_2m_2}$. Comparing the factors on the RHS of (3.4) with that of (3.2), we have the coupling coefficients

$$\begin{bmatrix}[1] & [n-1] & \Gamma\\ [\![n_1]\!]\ [\![n_2]\!] & [\![n_1']\!]\ [\![n_2']\!] & \Gamma_1\ \Gamma_2\end{bmatrix} \tag{3.5}$$

as usual, presented in table 2. It shows that the definition (2.2) and the derivation are consistent and reasonable. We also obtain the U(N) reduced matrix element of $b^\dagger_{lm}$:

$$\langle n\|b^\dagger\|n-1\rangle=\sqrt{n}. \tag{3.6}$$

Table 2. $\begin{bmatrix} [n-1] & [1] & \Gamma \\ [\![n_1']\!] \; [\![n_2']\!] & [\![n_1'']\!] \; [\![n_2'']\!] & \Gamma_1 \; \Gamma_2 \end{bmatrix}$ of $U(N) \supset U(N_1) \otimes U(N_2)$.

$[\![n_1']\!][\![n_2']\!]$	$[\![n_1'']\!][\![n_2'']\!]$ \ Γ; $\Gamma_1\ \Gamma_2$	$[n]$ $[\![n_1]\!]\ [\![n_2]\!]$	$[n-1\ 1]$ $[\![n_1]\!]\ [\![n_2]\!]$	$[n-1\ 1]$ $[\![n_1-1\ 1]\!]\ [\![n_2]\!]$	$[n-1\ 1]$ $[\![n_1]\!]\ [\![n_2-1\ 1]\!]$
$[\![n_1-1]\!][\![n_2]\!]$	$[\![1]\!][\![0]\!]$	$\left(\frac{n_1}{n}\right)^{1/2}$	$\left(\frac{n_2}{n}\right)^{1/2}$	1	0
$[\![n_1]\!][\![n_2-1]\!]$	$[\![0]\!][\![1]\!]$	$\left(\frac{n_2}{n}\right)^{1/2}$	$-\left(\frac{n_1}{n}\right)^{1/2}$	0	1

4. The state vectors classified by group chain II

Chain II provides another set of quantum numbers. One of them is σ, the generalised seniority, labelling IR of $O(N)$. Thus the state vector $|n\,\alpha\,L\,M\rangle$ can be expressed as

$$|n\ \overbrace{\sigma\ \sigma_1\alpha_1L_1\ \sigma_2\alpha_2L_2}^{\alpha}\ L\ M\rangle = \left(\frac{(2\sigma+N-2)!!}{\rho!(2\sigma+2\rho+N-2)!!}\right)^{1/2} P^{\dagger\rho}|\sigma\ \sigma\ \sigma_1\alpha_1L_1\ \sigma_2\alpha_2L_2\ L\ M\rangle \tag{4.1}$$

where $\rho=\frac{1}{2}(n-\sigma)$ and $P^\dagger=P_1^\dagger+P_2^\dagger$ is called the generalised pair creation operator, because

$$P_i^\dagger=\left(\frac{2l_i+1}{2}\right)^{1/2}(b_{l_i}^\dagger b_{l_i}^\dagger)_0^0 \qquad i=1,2$$

is the known l_i pair creation operator. The state vectors $|\sigma\,\sigma\,\sigma_1\alpha_1L_1\,\sigma_2\alpha_2\alpha_2\,L\,M\rangle$ satisfy

$$P|\sigma\ \sigma\ \sigma_1\alpha_1L_1\ \sigma_2\alpha_2L_2\ L\ M\rangle=0. \tag{4.2}$$

In order to solve (4.2) and get $|\sigma\,\sigma\,\sigma_1\alpha_1L_1\,\sigma_2\alpha_2L_2\,L\,M\rangle$, let us introduce a series of operators $Q^\dagger(2\Delta)$, $\Delta=1,2,\ldots,$

$$Q^\dagger(2\Delta)=\sum_{\delta_1+\delta_2=\Delta}(-1)^{\delta_2}\frac{(2\sigma_1+N_1-2)!!(2\sigma_2+N_2-2)!!}{\delta_1!(2\sigma_1+2\delta_1+N_1-2)!!\,\delta_2!(2\sigma_2+2\delta_2+N_2-2)!!}P_1^{\dagger\delta_1}P_2^{\dagger\delta_2}. \tag{4.3}$$

By straightforward derivation of $[PQ^\dagger(2\Delta)]$ we have

$$PQ^\dagger(2\Delta)|\sigma_1+\sigma_2\ \sigma_1\sigma_1\alpha_1L_1\ \sigma_2\sigma_2\alpha_2L_2\ L\ M\rangle=0. \tag{4.4}$$

Hence the states classified by chain II can be expressed in terms of the states classified by chain I, i.e.

$$|\sigma\ \sigma\ \sigma_1\alpha_1L_1\ \sigma_2\alpha_2L_2\ L\ M\rangle=DQ^\dagger(2\Delta)|\sigma_1+\sigma_2\ \sigma_1\sigma_1\alpha_1L_1\ \sigma_2\sigma_2\alpha_2L_2\ L\ M\rangle \tag{4.5}$$

where D is the normalised constant and is found by a tedious derivation

$$D=\left(\frac{\Delta!(2\sigma-2\Delta+N-4)!!(2\sigma_1+2\Delta+N_1-2)!!(2\sigma_2+2\Delta+N_2-2)}{(2\sigma+N-4)!!(2\sigma_1+N_1-2)!!(2\sigma_2+N_2-2)!!}\right)^{1/2}. \tag{4.6}$$

Equation (4.5) clearly shows that σ is the number of unpaired bosons in the O(N) level but may include Δ boson pairs in the O(N_1) and O(N_2) levels. Finally, we have the expression of state vectors classified by chain II in terms of the known one:

$$|n\ \sigma\ \sigma_1\alpha_1L_1\ \sigma_2\alpha_2L_2\ L\ M\rangle$$
$$=\left(\frac{\rho!\Delta!(2\sigma-2\Delta+N-4)!!(2\sigma+N-2)}{(2\sigma+2\rho+N-2)!!}\right)^{1/2}$$
$$\times\sum_{\substack{\omega_1+\omega_2=\rho\\ \rho_1+\rho_2=\rho+\Delta}}(-1)^{\rho_2-\omega_2}$$
$$\times\frac{(\rho_1!(2\sigma_1+2\rho_1+N_1-2)!!(2\sigma_1+2\Delta+N_1-2)!!\ \rho_2!(2\sigma_2+2\rho_2+N_2-2)!!(2\sigma_2+2\Delta+N_2-2)!!)^{1/2}}{\omega_1!(\rho_1-\omega_1)!(2\sigma_1+2\rho_1-2\omega_1+N_1-2)!!\ \omega_2!(\rho_2-\omega_2)!(2\sigma_2+2\rho_2-2\omega_2+N_2-2)!!}$$
$$\times|n\ \sigma_1+2\rho_1\ \sigma_1\alpha_1L_1\ \sigma_2+2\rho_2\ \sigma_2\alpha_2L_2\ L\ M\rangle \tag{4.7}$$

where $\rho=\frac{1}{2}(n-\sigma)$, $\sigma=\sigma_1+\sigma_2+2\Delta$. The following identities have been used in the above deviation:

$$\sum_{\delta=0}^{\Delta}\binom{\Delta}{\delta}\frac{(2\nu+1)!!(2\tau+1)!!}{(2\nu-2\delta+1)!!(2\tau+2\delta+1)!!}=\frac{(2\nu+2\tau+2\Delta+2)!!(2\tau+1)!!}{(2\nu+2\tau+2)!!(2\tau+2\Delta+1)!!} \tag{4.8a}$$

$$\sum_{\delta_1+\delta_2=\Delta}\frac{\Delta!(2\sigma-2\Delta+N-4)!!(2\sigma_1+2\Delta+N_1-2)!!(2\sigma_2+2\Delta+N_2-2)!!}{(2\sigma+N-4)!!\delta_1!(2\sigma_1+2\delta_1+N_1-2)!!\delta_2!(2\sigma_2+2\delta_2+N_2-2)!!}=1. \tag{4.8b}$$

5. CFP according to the group chain II

Matrix elements of $b^\dagger_{lm}$ in state $|n\,\sigma\,\sigma_1\alpha_1L_1\,\sigma_2\alpha_2L_2\,L\,M\rangle$ are constructed as before. First the components of the rank-1 tensor of U(N) are now labelled by chain II, i.e.

$$b^\dagger_{lm}=\begin{cases}b^\dagger_{l_1m_1}=b^\dagger_{[1]\ \langle1\rangle\ (1)\otimes(0)\ l_1\otimes0\ m_1\otimes0}\\ b^\dagger_{l_2m_2}=b^\dagger_{[1]\ \langle1\rangle\ (0)\otimes(1)\ 0\otimes l_2\ 0\otimes m_2}.\end{cases} \tag{5.1}$$

Second, using the generalised Wigner-Eckart theorem, we have

$$\langle n\ \sigma\ \sigma_1\alpha_1L_1\ \sigma_2\alpha_2L_2\ L\|b^\dagger_{l_1}\|n-1\ \sigma'\ \sigma_1'\alpha_1'L_1'\ \sigma_2\alpha_2L_2\ L'\rangle$$
$$=\langle n\|b^\dagger\|n-1\rangle\left[\begin{array}{cc|c}[1] & [n-1] & [n]\\ \langle1\rangle & \langle\sigma'\rangle & \langle\sigma\rangle\end{array}\right]$$
$$\times\left[\begin{array}{cc|c}\langle1\rangle & \langle\sigma'\rangle & \langle\sigma\rangle\\ (1)(0) & (\sigma_1')(\sigma_2) & (\sigma_1)(\sigma_2)\end{array}\right]\left[\begin{array}{cc|c}(1) & (\sigma_1') & (\sigma_1)\\ l_1 & \alpha_1'L_1' & \alpha_1L_1\end{array}\right]$$
$$\times\langle(l_1L_1')\ L_1\ L_2\ L|l_1\ (L_1'L_2)\ L'\ L\rangle. \tag{5.2}$$

All the factors on the RHS except

$$\left[\begin{array}{cc|c}\langle1\rangle & \langle\sigma'\rangle & \langle\sigma\rangle\\ (1)(0) & (\sigma_1')(\sigma_2) & (\sigma_1)(\sigma_2)\end{array}\right]$$

the coupling coefficients of O(N) ⊃ O(N_1)⊗O(N_2), are known. Similarly we may find the formula for $b^\dagger_{l_2m_2}$ also.

Table 3. $\left[\begin{matrix}\langle\sigma\rangle & \langle 1\rangle \\ (\sigma_1')(\sigma_2') & (\sigma_1'')(\sigma_2'')\end{matrix}\middle|\begin{matrix}\Gamma \\ \Gamma_1\ \Gamma_2\end{matrix}\right]$ of $SO(N)\supset SO(N_1)\otimes SO(N_2)$.

(*a*) $2\Delta=\sigma+1-\sigma_1-\sigma_2=0$

$(\sigma_1')(\sigma_2')$	$(\sigma_1'')(\sigma_2'')$ \ Γ, $\Gamma_1\ \Gamma_2$	$\langle\sigma+1\rangle$ $(\sigma_1)\ (\sigma_2)$	$\langle\sigma\ 1\rangle$ $(\sigma_1)\ (\sigma_2)$	$\langle\sigma\ 1\rangle$ $(\sigma_1-1\ 1)\ (\sigma_2)$	$\langle\sigma\ 1\rangle$ $(\sigma_1)\ (\sigma_2-1\ 1)$
$(\sigma_1-1)(\sigma_2)$	$(1)(0)$	$\left(\frac{\sigma_1}{\sigma+1}\right)^{1/2}$	$\left(\frac{\sigma_2}{\sigma+1}\right)^{1/2}$	1	0
$(\sigma_2)(\sigma_2-1)$	$(0)(1)$	$\left(\frac{\sigma_2}{\sigma+1}\right)^{1/2}$	$-\left(\frac{\sigma_1}{\sigma+1}\right)^{1/2}$	0	1

(*b*) $2\Delta=\sigma+1-\sigma_1-\sigma_2\neq 0$

$(\sigma_1')(\sigma_2')$	$(\sigma_1'')(\sigma_2'')$ \ Γ, $\Gamma_1\ \Gamma_2$	$\langle\sigma+1\ 0\ 0\rangle$ $(\sigma_1)\ (\sigma_2)$	$\langle\sigma-1\ 0\ 0\rangle$ $(\sigma_1)\ (\sigma_2)$
$(\sigma_1-1)(\sigma_2)$	$(1)(0)$	$\left(\frac{\sigma_1(2\sigma_1+2\Delta+N_1-2)(2\sigma-2\Delta+N-2)}{(2\sigma_1+N_1-2)(\sigma+1)(2\sigma+N-2)}\right)^{1/2}$	$\left(\frac{2\Delta\sigma_1(2\sigma_2+2\Delta+N_2-2)}{(2\sigma_1+N_1-2)(\sigma+N-3)(2\sigma+N-2)}\right)^{1/2}$
$(\sigma_1+1)(\sigma_2)$	$(1)(0)$	$\left(\frac{2\Delta(\sigma_1+N_1-2)(2\sigma_2+2\Delta+N_2-2)}{(2\sigma_1+N_1-2)(\sigma+1)(2\sigma+N-2)}\right)^{1/2}$	$\left(\frac{(\sigma_1+N_1-2)(2\sigma_1+2\Delta+N_1-2)(2\sigma-2\Delta+N-2)}{(2\sigma_1+N_1-2)(\sigma+N-3)(2\sigma+N-2)}\right)^{1/2}$
$(\sigma_1)(\sigma_2-1)$	$(0)(1)$	$\left(\frac{\sigma_2(2\sigma_2+2\Delta+N_2-2)(2\sigma-2\Delta+N-2)}{(2\sigma_2+N_2-2)(\sigma+1)(2\sigma+N-2)}\right)^{1/2}$	$-\left(\frac{2\Delta\sigma_2(2\sigma_1+2\Delta+N_1-2)}{(2\sigma_2+N_2-2)(\sigma+N-3)(2\sigma+N-2)}\right)^{1/2}$
$(\sigma_1)(\sigma_2+1)$	$(0)(1)$	$-\left(\frac{2\Delta(\sigma_2+N_2-2)(2\sigma_1+2\Delta+N_i-2)}{(2\sigma_2+N_2-2)(\sigma+1)(2\sigma+N-2)}\right)^{1/2}$	$-\left(\frac{(\sigma_2+N_2-2)(2\sigma_2+2\Delta+N_2-2)(2\sigma-2\Delta+N-2)}{(2\sigma_2+N_2-2)(\sigma+N-3)(2\sigma+N-2)}\right)^{1/2}$

$(\sigma_1')(\sigma_2')$	$(\sigma_1'')(\sigma_2'')$	$\langle\sigma\ 1\ 0\rangle$ $(\sigma_1)\ (\sigma_2)\ 1$	$\langle\sigma\ 1\ 0\rangle$ $(\sigma_1)\ (\sigma_2)\ 2$
$(\sigma_1-1)(\sigma_2)$	(1)(0)	$\left(\frac{2\Delta(\sigma_1+N_1-2)(\sigma_2+N_2-2)(2\sigma-2\Delta+N-2)}{(2\sigma_1+N_1-2)W_1(\sigma_1\sigma_2\sigma)}\right)^{1/2}$	$\left(\frac{(2\sigma_1+N_1-2)(\sigma_2+N_2-2)(2\sigma_2+N_2-2)\sigma_2}{(2\sigma_2+N_2-2)W_2(\sigma_1\sigma_2\sigma)}\right)^{1/2}$
$(\sigma_1+1)(\sigma_2)$	(1)(0)	$-\left(\frac{\sigma_1(2\sigma_1+2\Delta+N_1-2)(\sigma_2+N_2-2)(2\sigma_2+2\Delta+N_2-2)}{(2\sigma_1+N_1-2)W_1(\sigma_1\sigma_2\sigma)}\right)^{1/2}$	0
$(\sigma_1)(\sigma_2-1)$	(0)(1)	0	$-\left(\frac{\sigma_1(2\sigma_1+2\Delta+N_1-2)(\sigma_2+N_2-2)(2\sigma_2+2\Delta+N_2-2)}{(2\sigma_2+N_2-2)W_2(\sigma_1\sigma_2\sigma)}\right)^{1/2}$
$(\sigma_1)(\sigma_2+1)$	(0)(1)	$\left(\frac{\sigma_1(\sigma_1+N_1-2)(2\sigma_1+N_1-2)(2\sigma_2+N_2-2)}{(2\sigma_1+N_1-2)W_1(\sigma_1\sigma_2\sigma)}\right)^{1/2}$	$-\left(\frac{2\Delta\sigma_1\sigma_2(2\sigma-2\Delta+N-2)}{(2\sigma_2+N_2-2)W_2(\sigma_1\sigma_2\sigma)}\right)^{1/2}$

$(\sigma_1')(\sigma_2')$	$(\sigma_1'')(\sigma_2'')$	$\langle\sigma\ 1\ 0\rangle$ $(\sigma_1-1\ 1)\ (\sigma_2)$	$\langle\sigma\ 1\ 0\rangle$ $(\sigma_1+1\ 1)\ (\sigma_2)$	$\langle\sigma\ 1\ 0\rangle$ $(\sigma_1)\ (\sigma_2-1\ 1)$	$\langle\sigma\ 1\ 0\rangle$ $(\sigma_1)\ (\sigma_2+1\ 1)$
$(\sigma_1-1)(\sigma_2)$	(1)(0)	1	0	0	0
$(\sigma_1+1)(\sigma_2)$	(1)(0)	0	1	0	0
$(\sigma_1)(\sigma_2-1)$	(0)(1)	0	0	1	0
$(\sigma_1)(\sigma_2+1)$	(0)(1)	0	0	0	1

$W_1(\sigma_1\sigma_2\sigma)=\sigma_1(2\sigma_2+N_2-2)(\sigma_1+\sigma_2+N-4)+2\Delta(\sigma_2+N_2-2)(2\sigma-2\Delta+N-2).$

$W_2(\sigma_1\sigma_2\sigma)=(2\sigma_1+N_1-2)(\sigma_2+N_2-2)(\sigma_1+\sigma_2)+2\Delta(2\sigma-2\Delta+N-2).$

$$\left[\begin{array}{c|c}\langle\sigma\ 1\ 0\rangle & \langle\sigma\ 1\ 0\rangle\\ (\sigma_1)(\sigma_2)1 & (\sigma_1)(\sigma_2)2\end{array}\right]=\left(\frac{2\Delta(\sigma_1+N_1-2)(2\sigma-2\Delta+N-2)\sigma_2}{W_1(\sigma_1\sigma_2\sigma)W_2(\sigma_1\sigma_2\sigma)}\right)^{1/2}.$$

On the other hand, using the explicit expression (4.7), the reduced matrix elements of $b^\dagger_{lm}$ can be calculated straightforwardly. For example

$$\langle\sigma+1\ \ \sigma+1\ \ \sigma_1\alpha_1L_1\ \ \sigma_2\alpha_2L_2\ \ L\|b^\dagger_{l_1}\|\sigma\ \ \sigma\ \ \sigma'_1\alpha'_1L'_1\ \ \sigma_2\alpha_2L_2\ \ L'\rangle$$

$$=\left(\frac{\sigma_1(2\sigma_1+2\Delta+N_1-2)(2\sigma-2\Delta+N-2)}{(2\sigma_1+N_1-2)(2\sigma+N-2)}\right)^{1/2}$$

$$\times\left[\begin{array}{cc|c}(1) & (\sigma'_1) & (\sigma_1)\\ l_1 & \alpha'_1L'_1 & \alpha_1L_1\end{array}\right]\langle(l_1L'_1)\ L_1\ L_2\ L|l_1\ (L'_1L_2)\ L'\ L\rangle. \tag{5.3}$$

Comparing (5.3) with (5.2) we have the unknown coupling coefficients of $O(N)\supset O(N_1)\otimes O(N_2)$ presented in table 3. We finally obtain the formulae of CFP according to group chain II:

$$\langle n\ \sigma\ \sigma_1\alpha_1L_1\ \ \sigma_2\alpha_2L_2\ \ L\{|n-1\ \ \sigma'\ \ \sigma'_1\alpha'_1L'_1\ \ \sigma_2\alpha_2L_2\ \ L'\ \ l_1\ \ L\rangle$$

$$=\left[\begin{array}{cc|c}[1] & [n-1] & [n]\\ \langle 1\rangle & \langle\sigma'\rangle & \langle\sigma\rangle\end{array}\right]\left[\begin{array}{cc|c}\langle 1\rangle & \langle\sigma'\rangle & \langle\sigma\rangle\\ (1)(0) & (\sigma'_1)(\sigma_2) & (\sigma_1)(\sigma_2)\end{array}\right]$$

$$\times\left[\begin{array}{cc|c}(1) & (\sigma'_1) & (\sigma_1)\\ l_1 & \alpha'_1L'_1 & \alpha_1L_1\end{array}\right]\langle(l_1L'_1)\ L_1\ L_2\ L|l_1\ (L'_1L_2)\ L'\ L\rangle \tag{5.4a}$$

$$\langle n\ \sigma\ \sigma_1\alpha_1L_1\ \ \sigma_2\alpha_2L_2\ \ L\{|n-1\ \ \sigma'\ \ \sigma_1\alpha_1L_1\ \ \sigma'_2\alpha'_2L'_2\ \ L'\ \ l_2\ \ L\rangle$$

$$=\left[\begin{array}{cc|c}[1] & [n-1] & [n]\\ \langle 1\rangle & \langle\sigma'\rangle & \langle\sigma\rangle\end{array}\right]\left[\begin{array}{cc|c}\langle 1\rangle & \langle\sigma'\rangle & \langle\sigma\rangle\\ (0)(1) & (\sigma_1)(\sigma'_2) & (\sigma_1)(\sigma_2)\end{array}\right]$$

$$\times\left[\begin{array}{cc|c}(1) & (\sigma'_2) & (\sigma_2)\\ l_2 & \alpha'_2L'_2 & \alpha_2L_2\end{array}\right]\langle L_1\ (l_2L'_2)\ L_2\ L|l_2\ (L_1L'_2)\ L'\ L\rangle. \tag{5.4b}$$

The coefficients

$$\left[\begin{array}{cc|c}(1) & (\sigma'_i) & (\sigma_i)\\ l_i & \alpha'_iL'_i & \alpha_iL_i\end{array}\right]$$

are proportional to the CFP of a single-l_i boson system [6]. So are the CFP according to group chain II.

6. Summary

In this paper and the previous one [6] we have proposed an efficient procedure for the construction of CFP. Usually the computation involves a combination of numerical and analytic methods. The iterative parts in our formulae are squeezed, because the recursion relation depends on σ_i, not on n or σ, to avoid a lot of repetitive computations related to identical (σ_i) in different $\langle\sigma\rangle$ or $[n]$. Next, the generalised pair operator $P^\dagger$ and the series of operators $Q^\dagger(2\Delta)$ are easily extended through to multi-l boson or multi-j fermion systems.

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